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Topological representation of lattices and their homomorphisms

ALEKSANDER BŁASZCZYK

1. BASIC NOTIONS AND FACTS.

Results presented here were obtained jointly with Wojciech Bielas and will appear in [1].

An algebraic structure $\mathbb{L} = \langle L, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$, abbreviated $\mathbb{L}$, is called a *lattice* whenever the binary operations $\wedge$ and $\vee$ are commutative, associative, satisfy the absorption property and $x \wedge \mathbf{1} = x \vee \mathbf{0} = x$ holds for all $x \in L$.

A natural ordering in $\mathbb{L}$ is given by equivalences:

$$x \leq y \Longleftrightarrow x \wedge y = x \Longleftrightarrow x \vee y = y.$$

Then $\mathbf{0}$ is the smallest and $\mathbf{1}$ the greatest element. For a space X, $\mathbb{C}\mathrm{l}(X)$ denotes the lattice of all closed subsets of X, whereas $\mathcal{Z}(X)$ denotes the lattice of all zero-sets in X.

A lattice $\mathbb{L}$ is called:

(1) *distributive* if for all $x, y, z \in \mathbb{L}$ there is

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

(2) *normal* if it is distributive and for all $a, b \in \mathbb{L}$ with $a \wedge b = \mathbf{0}$ there exist $x, y \in \mathbb{L}$ such that

$$x \vee y = \mathbf{1} \text{ and } x \wedge a = y \wedge b = \mathbf{0},$$

(3) *separative* if it is distributive and for all $x, y \in \mathbb{L}$ with $x \nleq y$, there exists $z \in \mathbb{L} \setminus \{\mathbf{0}\}$ such that $z \leq x$ and $y \wedge z = \mathbf{0}$.

Let us note the following easy observations:

Fact 1.1. *Every Boolean lattice is a normal and separative lattice.*

Fact 1.2. *The lattice $\mathbb{C}\mathrm{l}(X)$ is normal iff the space X is normal.*

A family $\mathcal{L} \subseteq \mathbb{C}\mathrm{l}(X)$ is called a *closed base* in a space X whenever for every $F \in \mathbb{C}\mathrm{l}(X)$ there exists some $\mathcal{F} \subseteq \mathcal{L}$ such that $F = \bigcap \mathcal{F}$. Moreover, if $\mathcal{L}$ is closed under finite unions and finite intersections then it is called a *base lattice.*

Example 1.3. *If X is an infinite discrete space, then*

$$\mathbb{L} = \{F \subseteq X : |X \setminus F| < \omega\} \cup \{\emptyset\}$$

is a closed base for X but as a lattice it is not separative.

Let us leave without proof the following easy facts:

Proposition 1.4. *Let X be a compact Hausdorff space. If a sublattice $\mathbb{L} \subseteq \mathbb{C}\mathrm{l}(X)$ is a closed base for X, then the lattice $\mathbb{L}$ is both normal and separative.*

Proposition 1.5. *Let X be a Tychonoff space. Then the lattice $\mathcal{Z}(X)$ is both normal and separative.*

2. Ultrafilters

A nonempty set $\xi \subseteq \mathbb{L}$ is called *centered* provided that the following condition holds true:

$(*) \quad x_1, x_2, \ldots, x_n \in \xi \Rightarrow x_1 \wedge x_2 \wedge \ldots \wedge x_n > \mathbf{0}.$

The following fact is well known in the literature; see e.g. Koppelberg [6] or Sikorski [8].

Theorem 2.1 (Tarski's Theorem). *Every centered family is contained in a maximal one.*

For a lattice $\mathbb{L}$ we set

$$\mathrm{Ult}(\mathbb{L}) = \{\xi \subseteq \mathbb{L} \colon \xi \text{ is a maximal centered family}\}.$$

Elements of $\mathrm{Ult}(\mathbb{L})$ are called *ultrafilters* in the lattice $\mathbb{L}$. Directly from this definition we can obtain the following:

Lemma 2.2. *If $\mathbb{L}$ is a distributive lattice and $\xi \subseteq \mathbb{L}$ then $\xi \in \mathrm{Ult}(\mathbb{L})$ iff the following conditions hold true:*

(1) $\mathbf{0} \notin \xi$ *and* $\mathbf{1} \in \xi$,
(2) $x, y \in \xi \Rightarrow x \wedge y \in \xi$,
(3) $x \in \mathbb{L} \setminus \xi \Rightarrow (\exists y \in \xi)(x \wedge y = \mathbf{0})$,

for all $x, y \in \mathbb{L}$.

For a distributive lattice $\mathbb{L}$ the *Wallman topology* on $\mathrm{Ult}(\mathbb{L})$ is generated by the family

$$\{\mathrm{Ult}(\mathbb{L}) \setminus u^* \colon u \in \mathbb{L}\},$$

where $u^* = \{\xi \in \mathrm{Ult}(\mathbb{L}) \colon u \in \xi\}$.

The following theorem was proved first by Wallman [10]; see also Johnstone [5].

Theorem 2.3 (Wallman's Theorem). *If $\mathbb{L}$ is a distributive lattice, then the Wallman space $\mathrm{Ult}(\mathbb{L})$ is a compact T_1-space. If additionally the lattice $\mathbb{L}$ is normal, then $\mathrm{Ult}(\mathbb{L})$ is a compact Hausdorff space.*

Let us note that if $\mathbb{B}$ is a Boolean lattice, then the Wallman space $\mathrm{Ult}(\mathbb{B})$ coincide with the Stone space of $\mathbb{B}$. Also, if $\mathbb{L}$ is separative then it is isomorphic with the sublattice $\{u^* \colon u \in \mathbb{L}\}$ of $\mathbb{C}\mathrm{l}(\mathrm{Ult}(\mathbb{L}))$ and $\{u^* \colon u \in \mathbb{L}\}$ is a closed base for $\mathrm{Ult}(\mathbb{L})$. We have the following:

Theorem 2.4. *If the lattice $\mathbb{L} \subseteq \mathbb{C}\mathrm{l}(X)$ is a closed base for a compact Hausdorff space X, then $\mathrm{Ult}(\mathbb{L})$ is homeomorphic to X.*

Let us note the same compact Hausdorff space can be the Wallman space of several non-isomorphic lattices. To do this it is enough to consider for a compact space two closed bases of different size. In the theory of Boolean algebras the situation is completely different: every compact zero-dimensional space is the Stone space of the Boolean algebra consisting of all clopen subsets of the space and such a representation is unique.

3. Homomorphisms

It appears that, similarly like in the theory of Boolean algebras, homomorphisms of lattices appoints continuous functions of their Wallman spaces; see Johnstone [5], Simons [9] and also Kubiś [7]. We propose the following:

Theorem 3.1. *Let $\mathbb{K}, \mathbb{L}$ be normal lattices and let $\varphi : \mathbb{K} \to \mathbb{L}$ be a homomorphism. Then there exists a continuous function $\varphi^* : \mathrm{Ult}(\mathbb{L}) \to \mathrm{Ult}(\mathbb{K})$ given by the formula:*

$$\varphi^*(\xi) = \{x \in \mathbb{K} \colon x \wedge y > \mathbf{0} \text{ for all } y \in \varphi^{-1}[\xi]\}$$

for each $\xi \in \mathrm{Ult}(\mathbb{L})$.

The next theorem says that if **NLat** denotes the category of normal and distributive lattices with **0** and **1** and homomorphisms and **Comp** denotes the category of compact Hausdorff spaces and continuous mappings, then there exists a contravariant functor from **NLat** into **Comp**. This functor is also called the *Wallman functor.*

Theorem 3.2. *Assume* $\mathbb{K}$, $\mathbb{L}$, $\mathbb{M}$ *are normal lattices and let* $\varphi : \mathbb{K} \to \mathbb{L}$ *and* $\psi : \mathbb{L} \to \mathbb{M}$ *be homomorphisms. Then*

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*.$$

If $\mathrm{id}_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$ *is the identity, then* $(\mathrm{id}_{\mathbb{K}})^*$ *is the identity as well.*

Corollary 3.3. *If* $\varphi : \mathbb{K} \to \mathbb{L}$ *is an isomorphism, then* $\varphi^* : \mathrm{Ult}(\mathbb{L}) \to \mathrm{Ult}(\mathbb{K})$ *is a homeomorphism of Wallman spaces.*

Next theorem says that the Wallman functor described above carries monomorphisms into surjections.

Theorem 3.4. *If* $\mathbb{K}, \mathbb{L}$ *are normal lattices and* $\varphi : \mathbb{K} \to \mathbb{L}$ *is a monomorphism, then the function* $\varphi^* : \mathrm{Ult}(\mathbb{L}) \to \mathrm{Ult}(\mathbb{K})$ *is a continuous surjection.*

For a space X, $\mathbb{RC}(X)$ denotes the Boolean lattice (Boolean algebra) of all regular closed subsets of X. The operations in $\mathbb{RC}(X)$ are given by the formulas

(1) $F \vee G = F \cup G$,
(2) $F \wedge G = \mathrm{cl}\ \mathrm{Int}(F \cap G)$,
(3) $-F = \mathrm{cl}\,(X \setminus F)$

However, the Wallman functor does not carry epimorphisms into injections. The last property makes a difference with the Stone functor which carries epimorphisms of Boolean lattices onto embeddings of Stone spaces.

Example 3.5. If X is an infinite compact metric space then the homomorphism $h : \mathbb{C}\mathrm{l}(X) \to \mathbb{RC}(X)$ given by the formula

$$h(F) = \mathrm{cl}\ \mathrm{Int}\, F$$

is an epimorphism, but the function $h^* : \mathrm{Ult}(\mathbb{RC}(X)) \to \mathrm{Ult}(\mathbb{C}\mathrm{l}(X))$ is not one-to-one. In fact, since the lattice $\mathbb{RC}(X)$ is complete, the space $\mathrm{Ult}(\mathbb{RC}(X))$ is extremally disconnected and thus it cannot contain convergent sequences. On the other hand $\mathrm{Ult}(\mathbb{C}\mathrm{l}(X))$ is homeomorphic with X, hence it is a metric space.

4. Applications

We start with the following easy observation; see also Gillman and Jerison [4].

Proposition 4.1. *Let* X *be a Tychonoff space. If a separative normal sublattice* $\mathbb{L} \subseteq \mathbb{C}\mathrm{l}(X)$ *is a closed base in* X *and* $\mathcal{Z}(X) \subseteq \mathbb{L}$, *then* $\mathrm{Ult}(\mathbb{L})$ *is a compactification of* X. *Moreover,* $\mathrm{Ult}(\mathcal{Z}(X))$ *is homeomorphic to the Čech–Stone compactification of* X.

Let X be a compact Hausdorff space and let a lattice $\mathbb{L} \subseteq \mathbb{C}\mathrm{l}(X)$ be a closed base in X. Let $\mathbb{L}^c$ denotes the Boolean sublattice of $\mathcal{P}(X)$ generated by $\mathbb{L}$. Since $\mathbb{L}^c$ is a Boolean lattice the space $X^0(\mathbb{L}) = \mathrm{Ult}(\mathbb{L}^c)$ is a zero-dimensional compact space. Let $e \colon \mathbb{L} \to \mathbb{L}^c$ be the injection appointed by the inclusion $\mathbb{L} \subseteq \mathbb{L}^c$. Then, by the Theorem 3.4 we get a continuous surjection $e^* \colon \mathrm{Ult}(\mathbb{L}^c) \to \mathrm{Ult}(\mathbb{L})$. If $f_{X,\mathbb{L}} \colon \mathrm{Ult}(\mathbb{L}) \to X$ denotes the canonical homeomorphism (see Theorem 2.4), we set $p_{X,\mathbb{L}} = f_{X,\mathbb{L}} \circ e^*$.

Theorem 4.2. *Assume* X *and* Y *are compact Hausdorff and* $g \colon X \to Y$ *is a continuous map. If lattices* $\mathbb{L} \subseteq \mathbb{C}\mathrm{l}(X)$ *and* $\mathbb{K} \subseteq \mathbb{C}\mathrm{l}(Y)$ *are closed bases in* X *and* Y, *respectively, and* $g^{-1}[F] \in \mathbb{L}$ *for every* $F \in \mathbb{K}$ *then there exists a continuous map* $g^0 \colon X^0(\mathbb{L}) \to Y^0(\mathbb{K})$ *such that*

$$p_{Y,\mathbb{K}} \circ g^0 = g \circ p_{X,\mathbb{L}},$$

i.e. the following diagram is commutative:

$$\begin{tikzcd} X^0(\mathbb{L}) \arrow[r, "g^0"] \arrow[d, "p_{X,\mathbb{L}}"'] & Y^0(\mathbb{K}) \arrow[d, "p_{Y,\mathbb{K}}"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

A sublattice $\mathbb{L} \subseteq \mathbb{C}\mathrm{l}(X)$ is called *disjunctive*, if for all $x \in X$ and $F \in \mathbb{L}$ such that $x \notin F$, there is $G \in \mathbb{L}$ such that $x \in G$ and $F \cap G = \emptyset$.

Let us observed that if X is a T_1-space, then the lattice $\mathbb{C}\mathrm{l}(X)$ is disjunctive. But not every sublattice $\mathbb{L} \subseteq \mathbb{C}\mathrm{l}(X)$ has to be disjunctive, even if X is normal. However, we have the following:

Theorem 4.3 (Frink [2]). *If X is a T_1-space and there exists a disjunctive normal sublattice of $\mathbb{C}\mathrm{l}(X)$ which is a base in X, then X is a Tychonoff space.*

If X and Y are Tychonoff spaces then a bijection $\Phi : C(X) \to C(Y)$ of rings of continuous functions is a *ring isomorphism* whenever

$$\Phi(f+g) = \Phi(f) + \Phi(g) \text{ and } \Phi(f \cdot g) = \Phi(f) \cdot \Phi(g)$$

for all $f, g \in C(X)$. We have the following theorem:

Theorem 4.4. *If X and Y are Tychonoff spaces, and $C(X)$ and $C(Y)$ are ring isomorphic, then $\mathcal{Z}(X)$ and $\mathcal{Z}(Y)$ are isomorphic as lattices.*

As an immediate corollary we obtain the well known Gelfand–Kolmogoroff Theorem, see e.g. [4].

Corollary 4.5 (Gelfand–Kolmogoroff [3]). *If X and Y are compact Hausdorff spaces such that $C(X)$ is a ring isomorphic to $C(Y)$, then X is homeomorphic to Y.*

References

[1] W. Bielas, A. Błaszczyk, *Topological representation of lattice homomorphisms*, Topology and its Applications (to appear)

[2] O. Frink, *Compactifications and semi-normal spaces*, American Journal of Mathematics 86 (1964), 602–907.

[3] I. Gelfand, A. Kolmogoroff, *On rings of continuous functions on topological spaces*, Dokl. Akad. Nauk SSSR 22 (1939), 11–15.

[4] L. Gillman, M. Jerison, *Rings of Continuous Functions*, The University Series in Higher Mathematics, D. Van Nostrand Company Inc., Princeton (1960).

[5] P. T. Johnstone, *Stone spaces*, vol. 3 of Cambridge Studies in Advanced Mathematics, Cambridge (1982).

[6] S. Koppelberg, *Handbook of Boolean Algebras, vol. 1, General Theory of Boolean Algebras*, North-Holland (1989).

[7] W. Kubiś, *Compact spaces, lattices, and absoluteness: a survey*, arXiv:1402.1589v1 [math.GN].

[8] R. Sikorski, *Boolean algebras*, Springer-Verlag, Berlin (1960).

[9] H. Simmons, *Reticulated rings*, J. Algebra 66 (1980), 169–192.

[10] H. Wallman, *Lattices and topological spaces*, Annals of Mathematics 39 (1938), 112–126.

University of Silesia, Institute of Mathematics, Bankowa 14, 40-007 Katowice, Poland
E-mail address: `ablaszcz@math.us.edu.pl`